M339J: February 17th, 2023.

The parametric distribution: Definition

- "**Definition:**" A parametric distribution is a set of distribution functions where each of these distribution functions is fully specified through one or more (a **fixed and finite** number) parameters.
- All of the distributions in the Appendices A and B (in your tables) are parametric
- For individual examples, look at the problems we did so far in this course ...

The scale distribution: Definition

- **Definition:** A parametric distribution is a scale distribution if, when a random variable from that set of distributions is multiplied by a positive constant, the resulting random variable is also in that set of distributions.
- For instance, the Weibull distribution is a scale distribution
- Other examples are:

- exponential (see textbook),
- gamma (see textbook),
- normal (why?)

Example. [The Weibull Distribution]

$X \sim \text{Weibull}(\theta, \tau)$

$$\boxed{F_X(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\tau}}} \quad \checkmark$$

Claim. $\tilde{X} := \varkappa \cdot X$ is also Weibull.

$\varkappa > 0$

$y > 0$:

$$F_{\tilde{X}}(y) = \mathbb{P}[\tilde{X} \le y] = \mathbb{P}[\varkappa \cdot X \le y]$$

$\uparrow$ by def'n

$$= \mathbb{P}\left[X \le \frac{y}{\varkappa}\right]$$

$$= F_X\left(\frac{y}{\varkappa}\right)$$

$$= 1 - e^{-\left(\frac{y/\varkappa}{\theta}\right)^{\tau}}$$

$$= 1 - e^{-\left(\frac{y}{\varkappa \cdot \theta}\right)^{\tau}}$$

$$\Rightarrow \tilde{X} \sim \text{Weibull}(\tilde{\theta} = \varkappa \cdot \theta, \tilde{\tau} = \tau)$$

The scale distribution: Definition

- **Definition:** Let X be a random variable with nonnegative support which has a scale distribution.
  If a parameter of that scale distribution satisfies the following two conditions:

1. When a member of that scale distribution is multiplied by a positive constant, the scale parameter is multiplied by the **same** constant, while
2. all the other parameters remain the same,
   that parameter is called the scale parameter.

- For the exponential (we do have a scale parameter θ)
- For gamma (we again have a scale parameter θ)
- Weibull (again θ)
- For the lognormal we do not have a scale parameter (according to the parametrization used in the textbook) - although this is a scale distribution (why??)

Parametric distribution families

- **"Definition:"**
  A parametric distribution family is a set of parametric distributions that are related in some meaningful way.
- Most importantly, we can do the following:
- the set of parameters is finite - but we can decrease the exact number of parameters by setting some of them to be
- constant, e.g., exponential is a special type of gamma (how?)
- equal to each other, e.g., paralogistic is a special type of distribution from the transformed beta distribution family with $\tau = 1$ and $\alpha = \gamma$

Loss Elimination Ratio. (LER)

... is the ratio of the decrease in the insurer's expected pmt w/ an ordinary deductible d to the insurer's expected pmt w/ no deductible.

As usual: X ... loss random variable

Assume: $\mathbb{E}[X] < \infty$

$$LER = \frac{\mathbb{E}[X] - \mathbb{E}[(X-d)_+]}{\mathbb{E}[X]} = \boxed{\frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}}$$

Note: $\mathbb{E}[X \wedge d] \leq \mathbb{E}[X] \Rightarrow \boxed{LER \leq 1}$

Example. Let the ground-up loss r.v. X be exponential w/ mean 5000.

Assume that it's insured by an insurance policy w/ an ordinary deductible of 2500.

Find the loss elimination ratio!

$\rightarrow$: $X \sim$ Exponential (mean $= \theta = 5000$)

By def'n: $LER = \frac{\mathbb{E}[X \wedge d]}{\mathbb{E}[X]}$

$$LER = \frac{\cancel{\theta}(1 - e^{-\frac{d}{\theta}})}{\cancel{\theta}} = 1 - e^{-\frac{d}{\theta}} = F_X(d)$$

In this problem:

$$LER = 1 - e^{-\frac{2500}{5000}} = 1 - e^{-\frac{1}{2}} = \underline{0.3935}.$$

$\square$

89. You are given:

$X \sim$ Exponential (mean $= \theta$)

(i) Losses follow an exponential distribution with the same mean in all years.

(ii) The loss elimination ratio this year is 70%. $\boxed{LER = 0.7}$

(iii) The $\underbrace{\text{ordinary deductible for the coming year}}_{\tilde{d}}$ is 4/3 of $\overbrace{\text{the current deductible}}^{d}$.

$\tilde{d} = \frac{4}{3} d$

Calculate the loss elimination ratio for the coming year.

$\rightarrow$: $\widetilde{LER} = ?$

(A) 70%

(B) 75%

(C) 80%

(D) 85%

(E) 90%

(ii) $\Rightarrow 0.7 = 1 - e^{-\frac{d}{\theta}} \Rightarrow \boxed{e^{-\frac{d}{\theta}} = 0.3}$

$\widetilde{LER} = 1 - e^{-\frac{\tilde{d}}{\theta}} = 1 - e^{-\frac{\frac{4}{3}d}{\theta}} = 1 - \left(e^{-\frac{d}{\theta}}\right)^{4/3}$

$\widetilde{LER} = 1 - (0.3)^{4/3} = 0.799$ $\square$

90. Actuaries have modeled auto windshield claim frequencies. They have concluded that the number of windshield claims filed per year per driver follows the Poisson distribution with parameter λ, where λ follows the gamma distribution with mean 3 and variance 3.

Calculate the probability that a driver selected at random will file no more than 1 windshield claim next year.

(A) 0.15

(B) 0.19

(C) 0.20

(D) 0.24

(E) 0.31

126. The number of annual losses has a Poisson distribution with a mean of 5. The size of each loss has a two-parameter Pareto distribution with $\theta = 10$ and $\alpha = 2.5$. An insurance for the losses has an ordinary deductible of 5 per loss.

Calculate the expected value of the aggregate annual payments for this insurance.

(A) 8

(B) 13

(C) 18

(D) 23

(E) 28

$X \sim \text{Pareto}(\alpha=2, \theta=5)$

127. Losses in 2003 follow a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 5$. Losses in 2004 are uniformly 20% higher than in 2003. An insurance covers each loss subject to an ordinary deductible of 10. $\tilde{X} = 1.2X \sim ?$

Calculate the Loss Elimination Ratio in 2004.

(A) 5/9

$\boxed{\text{(B) } 5/8}$

(C) 2/3

(D) 3/4

(E) 4/5

$$\tilde{X} = \kappa \cdot X$$

$$y>0: \quad F_{\tilde{X}}(y) = \mathbb{P}[\kappa \cdot X \le y] = \mathbb{P}\left[X \le \frac{y}{\kappa}\right] = F_X\left(\frac{y}{\kappa}\right) = 1 - \left(\frac{\theta}{\frac{y}{\kappa}+\theta}\right)^{\alpha} = 1 - \left(\frac{\kappa\cdot\theta}{y+\kappa\cdot\theta}\right)^{\alpha}$$

128. DELETED

129. DELETED

$$\Rightarrow \tilde{X} \sim \text{Pareto}(\tilde{\alpha} = \alpha, \tilde{\theta} = \kappa\cdot\theta)$$

$\boxed{d=10}$

$$\widetilde{LER} = \frac{\mathbb{E}[\tilde{X}\wedge d]}{\mathbb{E}[\tilde{X}]} = \frac{\cancel{\frac{\tilde{\theta}}{\tilde{\alpha}-1}}\left(1-\left(\frac{\tilde{\theta}}{d+\tilde{\theta}}\right)^{\tilde{\alpha}-1}\right)}{\cancel{\frac{\tilde{\theta}}{\tilde{\alpha}-1}}} = 1 - \left(\frac{\tilde{\theta}}{d+\tilde{\theta}}\right)^{\tilde{\alpha}-1}$$

In this problem: $1 - \left(\frac{6}{10+6}\right)^{2-1} = 1 - \frac{6}{16} = 1 - \frac{3}{8} = \frac{5}{8}$ $\square$