# M339J: February 6th, 2023.

## Excess Loss (Random) Variable.

**Def'n.** Let $X$ be a (nonnegative) random variable. Let $d$ be a positive constant such that

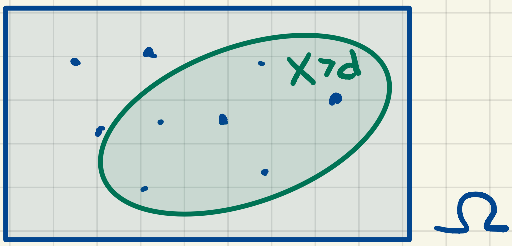
$$\boxed{\mathbb{P}[X>d]>0}$$

The excess loss random variables is usually denoted by $Y^P$ and it is defined as

$$Y^P = X-d \text{ given that } X>d$$

or

$$\boxed{Y^P = X-d \mid X>d}$$




It is also known as the per payment random variable.

**Note:** • All the values of $X$ less than $d$ are "discarded"

$\Rightarrow$ left truncated

• $d$ is subtracted $\Rightarrow$ shifted

**Def'n.** The mean excess loss function, denoted by $e_X(d)$, is defined as the expectation of $Y^P$, i.e.,

$$\boxed{e_X(d) = \mathbb{E}[X-d \mid \boxed{X>d}]}$$

By def'n,
$$e_X(d) = \frac{\mathbb{E}[(X-d)\cdot \mathbb{I}_{[X>d]}]}{\mathbb{P}[X>d]}$$

It's frequently convenient to write:

$$e_X(d) = \frac{\int_d^{+\infty} S_X(x)\,dx}{S_X(d)}$$

**Example.** Let $X \sim \text{Exponential}(\text{mean} = \theta)$.
Let $d > 0$.

**Note:** $\mathbb{P}[X > d] = S_X(d) = e^{-\frac{d}{\theta}} > 0$

**Define** $Y^P = X - d \mid X > d$

$$e_X(d) = \frac{\int_d^{+\infty} e^{-\frac{x}{\theta}}\,dx}{e^{-\frac{d}{\theta}}} = e^{\frac{d}{\theta}} \cdot (-\theta) e^{-\frac{x}{\theta}} \Big|_{x=d}^{+\infty}$$

$$= e^{\frac{d}{\theta}} (+\theta)\left(0 + e^{-\frac{d}{\theta}}\right) = \underline{\theta}$$

We could have simply used the memoryless property!

Let $X \sim \text{Exponential}(\text{mean} = \theta)$

What is the distribution of $Y^P$?

$Y^P \sim \text{Exponential}(\text{mean} = \theta)$

**Def'n.** The left censored and shifted random variable, usually denoted by $Y^L$ and called the per loss random variable is defined as:

$$\boxed{Y^L = \begin{cases} X - d & \text{if } X > d \\ 0 & \text{if } X \leq d \end{cases}}$$

Using the indicator random variables, we can write

$$\boxed{Y^L = (X - d) \cdot \mathbb{I}_{[X > d]}}$$

**Introduce:** the positive part function:

$$\xi \longmapsto (\xi)_+ := \begin{cases} \xi & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}$$

We can write $\boxed{Y^L = (X - d)_+}$

**Example.** Let $X \sim \text{Exponential}(\text{mean} = \theta)$.
What is the dist'n of the per loss random variable $Y^L$?

$\longrightarrow$: What is the support of $Y^L$?

$[0, +\infty)$

**Q**: What is the probability $\mathbb{P}[Y^L = 0]$?

$\longrightarrow$: $\mathbb{P}[Y^L = 0] = \mathbb{P}[X \leq d] = F_X(d) = \boxed{1 - e^{-\frac{d}{\theta}}} > 0$

Consider $y > 0$:

$F_{Y^L}(y) = \mathbb{P}[Y^L \leq y]$ (law of total probability)

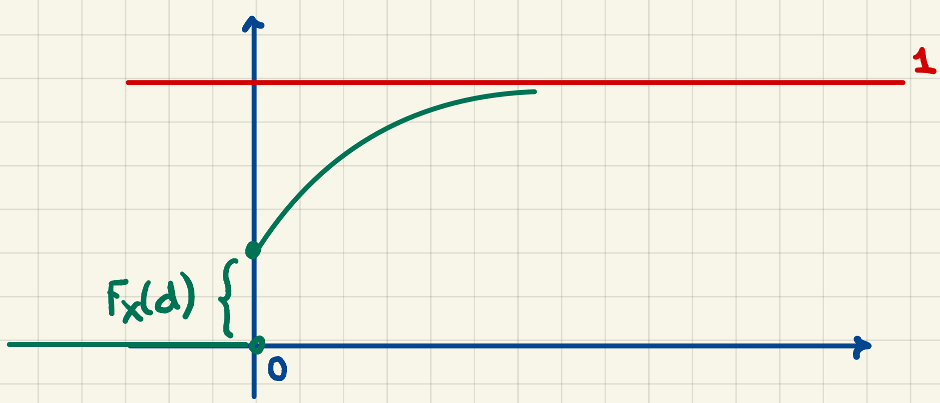
$= \mathbb{P}[Y^L \leq y, X \leq d] + \mathbb{P}[Y^L \leq y, X > d]$

$= \mathbb{P}[0 \leq y, X \leq d] + \mathbb{P}[X - d \leq y, X > d]$

$= \mathbb{P}[X \leq d] + \mathbb{P}[X \leq d + y, X > d]$

$= \mathbb{P}[X \leq d] + \mathbb{P}[d < X \leq d + y]$

$= \mathbb{P}[X \leq d + y] = F_X(d+y) = 1 - e^{-\frac{d+y}{\theta}}$




**Q**: Is it important that $X$ is exponential?

$\longrightarrow$: Not much ;)

**Q**: Assume that $X$ is continuous.
What type of a random variable is ...

$Y^P$ ... continuous

$Y^L$ ... mixed