M339 J: April 12th, 2023.

90. You are given the following observations on 185 small business policies:

Number of Claims	Number of Policies
0	80
1 or more	105

$1-\hat{q} = \frac{80}{185} = e^{-\lambda}$

$\hat{q} = \frac{105}{185}$ "success" prob. of 1 or more events

The number of claims per policy follows a Poisson distribution with parameter λ.

Using the maximum likelihood estimate of λ, determine the estimated probability of a policy having fewer than two claims.

(A) 0.79

(B) 0.84

(C) 0.89

(D) 0.95

(E) 0.98

→: The likelihood f'tion:

$$L(\lambda) = \Big(\underbrace{e^{-\lambda}\cdot\frac{\lambda^0}{0!}}_{e^{-\lambda}}\Big)^{80}\left(1-e^{-\lambda}\right)^{105}$$

The log-likelihood f'tion:

$$\ell(\lambda) = -80\lambda + 105\ln(1-e^{-\lambda})$$

$$\ell'(\lambda) = -80 + 105\cdot\frac{1}{1-e^{-\lambda}}(+1)(+1)e^{-\lambda} = 0$$

$$\frac{e^{-\lambda}}{1-e^{-\lambda}} = \frac{80}{105} = \frac{16}{21}$$

$$21e^{-\lambda} = 16 - 16e^{-\lambda}$$

$$37e^{-\lambda} = 16$$

$$\boxed{e^{-\lambda} = \frac{16}{37}} \Rightarrow \hat{\lambda} = \ln\left(\frac{37}{16}\right)$$

$$\mathbb{P}[X \le 1] = e^{-\lambda} + \lambda e^{-\lambda} = e^{-\lambda}(1+\lambda) = 0.795$$

$\square$

MLE: Negative Binomial.

$X \sim \text{NegBinomial}(r, \beta)$

pmf.
$$\boxed{f_X(x; r, \beta) = \binom{r+x-1}{x}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{x}}$$

$x_1, x_2, \dots, x_n$ are your observations

Likelihood f'tion:

$$L(r,\beta) = \prod_{j=1}^{n} f_X(x_j; r, \beta) = \boxed{\prod_{j=1}^{n} \binom{r+x_j-1}{x_j} \frac{\beta^{x_j}}{(1+\beta)^{r+x_j}}}$$

Log-likelihood f'tion:

$$\ell(r,\beta) = \sum_{j=1}^{n}\left(\ln\binom{r+x_j-1}{x_j} + x_j \cdot \ln(\beta) - (r+x_j)\ln(1+\beta)\right)$$

Method:
$$\frac{\partial}{\partial r}\ell(r,\beta) = \cdots = 0$$
$$\frac{\partial}{\partial \beta}\ell(r,\beta) = \cdots = 0$$

This is a system of two eq'ns w/ two unknowns.

$$\boxed{\hat{r}_{MLE}, \hat{\beta}_{MLE}}$$

An important feature: $\boxed{\hat{r}_{MLE} \cdot \hat{\beta}_{MLE} = \bar{x}}$

6. You are given:

Claim Size (X)	Number of Claims
(0, 25]	25
(25, 50]	28
(50, 100]	15
(100, 200]	6

Assume a uniform distribution of claim sizes within each interval.

Calculate $E(X^2) - E[(X \wedge 150)^2]$.

(A) Less than 200

(B) At least 200, but less than 300

(C) At least 300, but less than 400

(D) At least 400, but less than 500

(E) At least 500

7. The number of claims follows a negative binomial distribution with parameters β and r, where β is unknown and r is known. You wish to estimate β based on n observations, where $\bar{x}$ is the mean of these observations.

Determine the maximum likelihood estimate of β.

(A) $\bar{x} / r^2$

$\boxed{\text{(B)} \quad \bar{x} / r}$

(C) $\bar{x}$

(D) $r\bar{x}$

(E) $r^2\bar{x}$

$\rightarrow$: Because, r is known, we can omit it from the arguments.

$$L(\beta) = \prod_{j=1}^{n} \boxed{\binom{r+x_j-1}{x_j}} \frac{\beta^{x_j}}{(1+\beta)^{r+x_j}}$$

w/ r known, this is a multiplicative constant

$$L(\beta) \propto \frac{\beta^{\Sigma x_j}}{(1+\beta)^{r \cdot n + \Sigma x_j}}$$

$\Rightarrow$

$$\ell(\beta) \text{"="} (\Sigma x_j)\ln(\beta) - (r \cdot n + \Sigma x_j)\cdot \ln(1+\beta)$$

$$\ell'(\beta) = \frac{\cancel{n}\cdot\bar{x}}{\beta} - \frac{r\cdot\cancel{n} + \cancel{n}\cdot\bar{x}}{1+\beta} = 0$$

$$\bar{x}(1+\cancel{\beta}) = (r+\cancel{\bar{x}})\beta$$

$$\boxed{\hat{\beta}_{MLE} = \frac{\bar{x}}{r}}$$

$\square$

Problem. Annual claims follow a negative binomial distribution. The following claim count observations are available.

Year	Claim Counts
2005	0
2004	3
2003	5

Assuming that each year is independent, find the likelihood function of the sample.

$\rightarrow$:

$$L(r,\beta) = \overbrace{\boxed{\binom{r+0-1}{0}}}^{1}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{0}\cdot$$

$$\cdot\boxed{\binom{r+3-1}{3}}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{3}\cdot$$

$$\cdot\boxed{\binom{r+5-1}{5}}\left(\frac{1}{1+\beta}\right)^{r}\left(\frac{\beta}{1+\beta}\right)^{5}$$

$$L(r,\beta)=\boxed{\frac{(r+2)(r+1)r\,(r+4)(r+3)(r+2)(r+1)\cdot r}{3!\ \cdot 5!}}\left(\frac{1}{1+\beta}\right)^{3r}\left(\frac{\beta}{1+\beta}\right)^{8}$$

$$\frac{(r+4)(r+3)(r+2)^2(r+1)^2r^2}{3!\cdot 5!}$$

□

Problem. You are given:

(i) The number of claims follows a geometric dist'n w/ mean β.

(ii) Half of the observations of numbers of claims are 0.

Determine the maximum likelihood estimate of β.

→: $\hat{\beta}=?$

$\boxed{n \text{... sample size}}$

Method I.

$$L(\beta)=\left(\frac{1}{1+\beta}\right)^{\frac{n}{2}}\cdot\left(\frac{\beta}{1+\beta}\right)^{\frac{n}{2}}=\frac{\beta^{\frac{n}{2}}}{(1+\beta)^{n}}$$

$$\ell(\beta)=\frac{n}{2}\cdot\ln(\beta)-n\cdot\ln(1+\beta)$$

$$\ell'(\beta)=\frac{\cancel{n}}{2}\cdot\frac{1}{\beta}-\cancel{n}\cdot\frac{1}{1+\beta}=0$$

$$2\beta=1+\beta \Rightarrow \boxed{\hat{\beta}=1}$$

Method II. $\hat{q}=\widehat{\left(\frac{1}{1+\beta}\right)}=\frac{1}{2} \Rightarrow \boxed{\hat{\beta}=1}$

□