M339J: April 10th, 2023.

MLE: Bernoulli.

$X \sim \text{Bernoulli}(q)$ ← the probab. of success

$\text{Support}(X) = \{0, 1\}$

$$X \sim \begin{cases} 0 & \text{w/ probab. } 1-q \\ 1 & \text{w/ probab. } q \end{cases}$$

Let $x_1, x_2, \dots, x_n$ be the observations from Bernoulli(q).

They will all be 0 or 1.

We write the pmf as:

$$f(x;q) = \begin{cases} q & \text{if } x=1 \\ 1-q & \text{if } x=0 \end{cases}$$

$$\boxed{f(x;q) = q^x \cdot (1-q)^{1-x}} \quad \checkmark$$

The likelihood f'tion:

$$L(q) = \prod_{j=1}^{n} f_X(x_j;q) = \prod_{j=1}^{n} \left(q^{x_j} (1-q)^{1-x_j} \right)$$

$$L(q) = \prod_{j=1}^{n} q^{x_j} \cdot \prod_{j=1}^{n} (1-q)^{1-x_j} = q^{\sum x_j} (1-q)^{n-\sum x_j}$$

The log-likelihood f'tion:

$$\ell(q) = \left(\sum x_j\right) \cdot \ln(q) + \left(n - \sum x_j\right) \cdot \ln(1-q)$$

Differentiating w.r.t. q, we get

$$\ell'(q) = \left(\sum x_j\right) \cdot \frac{1}{q} + \left(n - \sum x_j\right) \frac{1}{1-q} (-1) = 0$$

$$\frac{\sum x_j}{q} = \frac{n - \sum x_j}{1-q}$$

We solve for q in:

$$\left(\sum x_j\right)(1-\cancel{q}) = \left(n - \cancel{\sum x_j}\right) \cdot q$$

$$\boxed{\hat{q}_{MLE} = \frac{\sum x_j}{n} = \bar{x}}$$

sample mean, i.e., the proportion of successes

Problem. When Mr. Jones visits his local race track, he places three independent bets. In his last 20 visits,
he lost all of his bets 10 times,
won one bet 7 times,
and won two bets three times.
He has never won all three of his bets.
Find the maximum likelihood estimate of the probability that Mr. Jones wins an individual bet.

$\longrightarrow$: Each bet $\sim$ Bernoulli(q)

$\Rightarrow \quad \hat{q} = \frac{13}{60}$ □

MLE: Poisson.

$X \sim \text{Poisson}(\text{mean} = \lambda)$

pmf: $f_X(x; \lambda) = e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$

Let $x_1, x_2, \ldots, x_n$ be the observations from Poisson(λ).

Then, the likelihood f'tion is:

$$L(\lambda) = \prod_{j=1}^{n} f_X(x_j; \lambda) = \prod_{j=1}^{n} \left(e^{-\lambda} \frac{\lambda^{x_j}}{x_j!} \right)$$

$$L(\lambda) = e^{-n \cdot \lambda} \cdot \prod_{j=1}^{n} \lambda^{x_j} \prod_{j=1}^{n} \frac{1}{x_j!} \propto e^{-n \cdot \lambda} \lambda^{\sum x_j}$$

The log-likelihood f'tion is:

$$\ell(\lambda) = -n \cdot \lambda + \left(\sum x_j\right) \ln(\lambda)$$

$$\Rightarrow \ell'(\lambda) = -n + \left(\sum x_j\right) \cdot \frac{1}{\lambda} = 0$$

$$\boxed{\hat{\lambda}_{MLE} = \frac{\sum x_j}{n} = \bar{x}}$$

sample mean

Problem. Let claim frequency in a particular year be Poisson(λ).
Two claims were observed in risk Year #1.
One claim was observed in risk Year #2.
The Years are independent.
If λ is known to be an integer, what is its maximum likelihood estimate?

$\rightarrow$: $X \sim \text{Poisson}(\lambda)$

By our formula: $\hat{\lambda} = \frac{2+1}{2} = 1.5$

$\Rightarrow$ The two candidates for the answer: 1 and 2.

$$L(1) = e^{-1} \cdot \frac{1}{2} \cdot e^{-1} = e^{-2} \cdot \frac{1}{2} = 0.0677$$

$$L(2) = e^{-2} \cdot \frac{2^{\cancel{2}}}{\cancel{2}} \cdot e^{-2} \cdot \frac{2}{1} = 4e^{-4} = 0.0733$$

$\square$

39. You are given:

(i) A sample of losses is:

600 700 900

(ii) No information is available about losses of 500 or less.

(iii) Losses are assumed to follow an exponential distribution with mean θ.

Calculate the maximum likelihood estimate of θ.

(A) 233

(B) 400

(C) 500

(D) 733

(E) 1233

40. You are given:

(i) The number of claims follows a Poisson distribution with mean λ.

(ii) Observations other than 0 and 1 have been deleted from the data.

(iii) The data contain an equal number of observations of 0 and 1.

Condition on all the remaining data pts being ≤ 1.

Calculate the maximum likelihood estimate of λ.

(A) 0.50

(B) 0.75

(C) 1.00

(D) 1.25

(E) 1.50

$$L(\lambda) = \frac{1}{(\mathbb{P}[X \leq 1; \lambda])^n} \underbrace{\left(e^{-\lambda} \cdot \frac{\lambda^0}{0!}\right)}_{e^{-\lambda}}{}^{\frac{n}{2}} \underbrace{\left(e^{-\lambda} \cdot \frac{\lambda^1}{1!}\right)}_{\lambda e^{-\lambda}}{}^{n/2}$$

$$L(\lambda) = \frac{1}{(\cancel{e^{-\lambda}}(1+\lambda))^n} \cdot \cancel{e^{-\lambda \cdot \frac{n}{2}}} \cdot \lambda^{\frac{n}{2}} \cdot \cancel{e^{-\lambda \cdot \frac{n}{2}}}$$

$$L(\lambda) = \frac{\lambda^{\frac{n}{2}}}{(1+\lambda)^n}$$

$$\Rightarrow \ell(\lambda) = \frac{n}{2} \cdot \ln(\lambda) - n \cdot \ln(1+\lambda)$$

$$\Rightarrow \ell'(\lambda) = \frac{n}{2} \cdot \frac{1}{\lambda} - n \cdot \frac{1}{1+\lambda} = 0$$

$$\frac{\cancel{n}}{2\lambda} = \frac{\cancel{n}}{1+\lambda}$$

$$2 \cdot \lambda = 1 + \lambda \Rightarrow \boxed{\hat{\lambda} = 1}$$

$\square$