M339J: April 5th, 2023.

276. For a group of policies, you are given:

(i) Losses follow the distribution function

$$\boxed{F(x)} = 1 - \theta / x, \quad x > 0.$$

(ii) A sample of 20 losses resulted in the following:

Interval	Number of Losses
$\boxed{(0,10]}$	9
$\boxed{(10, 25]}$	6
$\boxed{(25, \infty)}$	5

Calculate the maximum likelihood estimate of θ.

→: Likelihood Function:

(A) 5.00

(B) 5.50

(C) 5.75

(D) 6.00

(E) 6.25

$$L(\theta) = (F(10))^9 (F(25) - F(10))^6 (1 - F(25))^5$$

$$= \left(1 - \frac{\theta}{10}\right)^9 \left(\cancel{1} - \frac{\theta}{25} - \left(\cancel{1} - \frac{\theta}{10}\right)\right)^6 \left(\cancel{1} - \left(\cancel{1} - \frac{\theta}{25}\right)\right)^5$$

$$= \left(1 - \frac{\theta}{10}\right)^9 \theta^6 \left(\frac{1}{10} - \frac{1}{25}\right)^6 \cdot \frac{\theta^5}{25^5}$$

$$= \frac{(10-\theta)^9}{10^9} \theta^{11} \left(\frac{1}{10} - \frac{1}{25}\right)^6 \frac{1}{25^5}$$

$$L(\theta) \underset{\text{proportional to}}{\underset{\uparrow}{\propto}} (10-\theta)^9 \cdot \theta^{11}$$

⇒ Up to a constant, our log-likelihood is

$$\ell(\theta) \text{"="} 9 \cdot \ln(10-\theta) + 11 \cdot \ln(\theta)$$

$$\Rightarrow \quad \ell'(\theta) = 9 \cdot (-1) \cdot \frac{1}{10-\theta} + 11 \cdot \frac{1}{\theta} = 0$$

$$\frac{11}{\theta} = \frac{9}{10-\theta}$$

$$110 - 11\theta = 9\theta$$

$$20\theta = 110$$

$$\hat{\theta}_{MLE} = 5.5 \quad \square$$

MLE: Censored and truncated data.

Censoring.

If the j^{th} data point is censored @ u_j,

then we look @ the probability of "hitting" $[u_j, +\infty)$.

Truncation.

If the j^{th} data point is truncated @ d_j,

then we can do:

- **shifting**: Let x_j be the value of the actual loss; construct $L(\theta)$ based on the actual losses w/ the model for the actual losses;
- **conditioning**: omit $x_j < d_j$;

for the remainder

$$L(\theta) = \prod_{j=1}^{n} \frac{f_X(x_j; \theta)}{S_X(d_j; \theta)}$$

for individual obsevations.

218. The random variable X has survival function:

$$S_X(x) = \frac{\theta^4}{(\theta^2 + x^2)^2} = \theta^4 (\theta^2 + x^2)^{-2}$$

Two values of X are observed to be 2 and 4. One other value exceeds 4.

x_1 x_2 $\boxed{u_3 = 4}$

Calculate the maximum likelihood estimate of θ.

$\rightarrow$: $L(\theta) = f(2) \cdot f(4) \cdot S(4)$

$f(x) = -S'(x) = +\theta^4 (+2)(\theta^2+x^2)^{-3} \cdot (2x)$

$f(x) = \dfrac{4\theta^4 \cdot x}{(\theta^2+x^2)^3}$

(A) Less than 4.0

(B) At least 4.0, but less than 4.5

(C) At least 4.5, but less than 5.0

(D) At least 5.0, but less than 5.5

(E) At least 5.5

219. For a portfolio of policies, you are given:

(i) The annual claim amount on a policy has probability density function:

$$f(x \mid \theta) = \frac{2x}{\theta^2}, \quad 0 < x < \theta$$

(ii) The prior distribution of θ has density function:

$$\pi(\theta) = 4\theta^3, \quad 0 < \theta < 1$$

(iii) A randomly selected policy had claim amount 0.1 in Year 1.

Calculate the Bühlmann credibility estimate of the claim amount for the selected policy in Year 2.

(A) 0.43

(B) 0.45

(C) 0.50

(D) 0.53

(E) 0.56

Our likelihood:

$$L(\theta) = \frac{4\cdot\theta^4\cdot 2}{(\theta^2+4)^3}\cdot\frac{4\cdot\theta^4\cdot 4}{(\theta^2+16)^3}\cdot\frac{\theta^4}{(\theta^2+16)^2} \propto \frac{\theta^{12}}{(\theta^2+4)^3(\theta^2+16)^5}$$

$$\Rightarrow \quad \ell(\theta) \text{"="}\, 12\cdot\ln(\theta) - 3\cdot\ln(\theta^2+4) - 5\ln(\theta^2+16)$$

$$\Rightarrow \quad \ell'(\theta) = 12\cdot\frac{1}{\theta} - 3\cdot\frac{1}{\theta^2+4}(2\theta) - 5\cdot\frac{1}{\theta^2+16}(2\cdot\theta) = 0$$

$$\frac{12}{\theta} = \frac{6\theta}{\theta^2+4} + \frac{10\theta}{\theta^2+16}$$

$$12(\theta^2+4)(\theta^2+16) = 6\theta^2(\theta^2+16) + 10\theta^2(\theta^2+4)$$

$$12\theta^4 + 12\cdot 16\cdot\theta^2 + 12\cdot 4\cdot\theta^2 + \underline{12\cdot 4\cdot 16} = 6\theta^4 + 6\cdot 16\cdot\theta^2 + 10\theta^4 + 40\theta^2$$

$$4\theta^4 - 104\theta^2 - 768 = 0 \quad /:4$$

$$\theta^4 - 26\theta^2 - 192 = 0$$

$$\theta^2_{1,2} = \frac{26 \pm\sqrt{26^2+4\cdot 192}}{2} = 32 \Rightarrow \hat{\theta}_{MLE} = \sqrt{32} = 5.6569 \quad \square$$

Example. Truncated data w/ a common d.

The likelihood f'tion:

$$L(\theta) = \prod_{j=1}^{n} \frac{f_X(x_j;\theta)}{S_X(d;\theta)} = \frac{1}{(S_X(d;\theta))^n}\cdot\prod_{j=1}^{n} f_X(x_j;\theta)$$

$\Rightarrow$ the log-likelihood f'tion:

$$\ell(\theta) = \sum_{j=1}^{n}\ln(f_X(x_j;\theta)) - n\ln(S_X(d;\theta))$$