M339J: April 3rd, 2023.

35. You are given the following information about a credibility model:

First Observation	Unconditional Probability	Bayesian Estimate of Second Observation
1	1/3	1.50
2	1/3	1.50
3	1/3	3.00

Calculate the Bühlmann credibility estimate of the second observation, given that the first observation is 1.

(A) 0.75

(B) 1.00

(C) 1.25

(D) 1.50

(E) 1.75

36. DELETED

37. A random sample of three claims from a dental insurance plan is given below:

x_1 x_2 x_3

225 525 950

Claims are assumed to follow a Pareto distribution with parameters $\theta = 150$ and α.

Calculate the maximum likelihood estimate of α.

$n=3$

$$\hat{\alpha}_{MLE} = \frac{3}{\ln(375)+\ln(675)+\ln(1100)-3\ln(150)}$$

$\hat{\alpha}_{MLE} = 0.6798$ □

(A) Less than 0.6

(B) At least 0.6, but less than 0.7

(C) At least 0.7, but less than 0.8

(D) At least 0.8, but less than 0.9

(E) At least 0.9

Grouped Data.

The data are grouped into "bins".

$$c_0 < c_1 < \cdots < c_k$$

We are given the number of observations in each interval:

$$(c_{j-1}, c_j].$$

Denote this number of observations by n_j for each $j = 1, \ldots, k$.

Then, $n_1 + n_2 + \cdots + n_k = \underline{n}$, i.e., the sample size.

For each "bin":

$$\left(\mathbb{P}[c_{j-1} < X \le c_j]\right)^{n_j} =$$

$$= \left(\mathbb{P}[X \le c_j] - \mathbb{P}[X \le c_{j-1}]\right)^{n_j}$$

$$= \left(F_X(c_j) - F_X(c_{j-1})\right)^{n_j}$$

The overall likelihood function is:

$$\boxed{L(\theta) = \prod_{j=1}^{k} \left(F_X(c_j;\theta) - F_X(c_{j-1};\theta)\right)^{n_j}}$$

Then, the loglikelihood function is:

$$\boxed{\ell(\theta) = \sum_{j=1}^{k} n_j \cdot \ln\left(F_X(c_j;\theta) - F_X(c_{j-1};\theta)\right)}$$

56. You are given the following information about a group of policies:

Claim Payment		Policy Limit		
5	<	50	unmodified	$x_1 = 5$
15	<	50	unmodified	$x_2 = 15$
60	<	100	unmodified	$x_3 = 60$
100	=	100	censoring	$(100, +\infty)$
500	=	500	censoring	$(500, +\infty)$
500	<	1000	unmodified	$x_4 = 500$

Determine the likelihood function.

(A) $f(50)f(50)f(100)f(100)f(500)f(1000)$

(B) $f(50)f(50)f(100)f(100)f(500)f(1000)/[1-F(1000)]$

(C) $f(5)f(15)f(60)f(100)f(500)f(500)$

(D) $f(5)f(15)f(60)f(100)f(500)f(1000)/[1-F(1000)]$

(E) $f(5)f(15)f(60)[1-F(100)][1-F(500)]f(500)$

43. You are given:

(i) The prior distribution of the parameter Θ has probability density function:

$$\pi(\theta) = \frac{1}{\theta^2}, \quad 1 < \theta < \infty$$

(ii) Given $\Theta = \theta$, claim sizes follow a Pareto distribution with parameters $\alpha = 2$ and θ.

A claim of 3 is observed.

Calculate the posterior probability that Θ exceeds 2.

(A) 0.33

(B) 0.42

(C) 0.50

(D) 0.58

(E) 0.64

44. You are given:

(i) Losses follow an exponential distribution with mean θ. $X \sim \text{Exponential}(\theta)$

(ii) A random sample of 20 losses is distributed as follows:

Loss Range	Frequency
$\boxed{[0, 1000]}$	7
(1000, 2000]	6
$\boxed{(2000, \infty)}$	7

Calculate the maximum likelihood estimate of θ.

$\longrightarrow$: $F_X(x;\theta) = 1 - e^{-\frac{x}{\theta}}$

(A) Less than 1950

(B) At least 1950, but less than 2100

(C) At least 2100, but less than 2250

(D) At least 2250, but less than 2400

(E) At least 2400

Our likelihood function:

$$L(\theta) = \left(1 - e^{-\frac{1000}{\theta}}\right)^7 \left[\left(\cancel{1} - e^{-\frac{2000}{\theta}}\right) - \left(\cancel{1} - e^{-\frac{1000}{\theta}}\right)\right]^6 \left(e^{-\frac{2000}{\theta}}\right)^7$$

$$L(\theta) = \left(1 - e^{-\frac{1000}{\theta}}\right)^7 \left(e^{-\frac{1000}{\theta}} - e^{-\frac{2000}{\theta}}\right)^6 \left(e^{-\frac{2000}{\theta}}\right)^7$$

Substitute $\boxed{y = e^{-\frac{1000}{\theta}}}$ (a strictly increasing transform)

$$L(y) = (1-y)^7 (y - y^2)^6 (y^2)^7 = (1-y)^7 y^6 (1-y)^6 y^{14}$$

$$L(y) = y^{20}(1-y)^{13}$$

The loglikelihood: $\ell(y) = 20\ln(y) + 13\ln(1-y)$

$$\ell'(y) = 20 \cdot \frac{1}{y} + 13(-1) \cdot \frac{1}{1-y} = 0$$

$$\frac{20}{y} = \frac{13}{1-y}$$

$$20 - 20y = 13y$$

$$33y = 20$$

$$\boxed{y = \frac{20}{33}}$$

Finally, we substitute back:

$$\frac{20}{33} = e^{-\frac{1000}{\theta}}$$

$$\ln\left(\frac{20}{33}\right) = -\frac{1000}{\theta}$$

$$\hat{\theta}_{MLE} = \frac{1000}{\ln\left(\frac{33}{20}\right)} = 1996.9 \quad \square$$

276. For a group of policies, you are given:

(i) Losses follow the distribution function

$$F(x) = 1 - \theta / x, \quad x > 0.$$

(ii) A sample of 20 losses resulted in the following:

Interval	Number of Losses
(0,10]	9
(10, 25]	6
$(25, \infty)$	5

Calculate the maximum likelihood estimate of θ.

(A) 5.00

(B) 5.50

(C) 5.75

(D) 6.00

(E) 6.25