The University of Texas at Austin
In-Class Work 2

M339G Predictive Analytics

September 17, 2026

From linear algebra towards multiple linear regression.

Problem 2.1: Defining a hyperplane
In $\mathbb{R}^3$, a hyperplane is a 2-dimensional flat surface defined by an equation of the form

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

Write the equation of the hyperplane through the point $(1, 2, 3)$ with normal vector $\boldsymbol{a} = (2, -1, 4)$.

Solution: *We substitute the point into $\boldsymbol{a} \cdot \boldsymbol{x} = b$, to get*

$$2(1) - 1(2) + 4(3) = 2 - 2 + 12 = 12$$

So the hyperplane is

$$2x_1 - x_2 + 4x_3 = 12$$

Problem 2.2: Normal vector and orientation
For the hyperplane $3x_1 + 4x_2 - 5x_3 = 10$, identify the normal vector, and determine whether the point $(2, 1, 0)$ lies on the hyperplane, above it, or below it (i.e., compute $\boldsymbol{a} \cdot \boldsymbol{x} - b$ and interpret its sign).

Solution: *The normal vector is $\boldsymbol{a} = (3, 4, -5)$. At $(2, 1, 0)$:*

$$3(2) + 4(1) - 5(0) = 10$$

so $\boldsymbol{a} \cdot \boldsymbol{x} - b = 0$ — the point lies exactly on the hyperplane.

Problem 2.3: From a geometric hyperplane to a linear function
A hyperplane in $\mathbb{R}^n$ can be written as the graph of a function

$$x_n = \beta_0 + \beta_1x_1 + \cdots + \beta_{n-1}x_{n-1}$$

by solving $\boldsymbol{a} \cdot \boldsymbol{x} = b$ for the last coordinate. Rewrite the hyperplane $2x_1 - 3x_2 + x_3 = 6$ (in $\mathbb{R}^3$) in the form $x_3 = \beta_0 + \beta_1x_1 + \beta_2x_2$, and identify $\beta_0, \beta_1, \beta_2$.

Solution:

$$x_3 = 6 - 2x_1 + 3x_2$$

so $\beta_0 = 6$, $\beta_1 = -2$, $\beta_2 = 3$. This is exactly the shape of a regression equation, i.e., one variable expressed as an affine combination of the others.

PROBLEM 2.4: Relabeling as a regression model

Suppose we now treat x_3 from the previous problem as a response variable y, and x_1, x_2 as predictors, but real data won't lie exactly on a hyperplane. Write the multiple linear regression model for one observation i, using coefficients $\beta_0, \beta_1, \beta_2$ and an error term ε_i, based on that same structural form.

SOLUTION:

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

*Each observed y_i is a point near — **not on** — the hyperplane $\beta_0 + \beta_1 x_1 + \beta_2 x_2$ in (x_1, x_2, y) space, displaced vertically by ε_i.*

PROBLEM 2.5: The regression hyperplane as a best fit

Given the model

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

for n observations, explain in matrix form how the hyperplane's coefficients are chosen, and write the normal equations that determine $\hat{\beta}$.

SOLUTION: *Stack the data into $\boldsymbol{y}$ ($n \times 1$), design matrix $\boldsymbol{X}$ ($n \times 3$, with a leading column of 1's for β_0), and coefficient vector $\boldsymbol{\beta}$ (3×1), giving*

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$$

Least squares chooses $\hat{\boldsymbol{\beta}}$ to minimize $\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\|^2$, the sum of squared vertical distances from the data points to the hyperplane. Setting the gradient to zero gives the normal equations:

$$\boldsymbol{X}^T \boldsymbol{X} \hat{\boldsymbol{\beta}} = \boldsymbol{X}^T \boldsymbol{y}$$

so

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}^T \boldsymbol{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{y}$$

when $\boldsymbol{X}^T \boldsymbol{X}$ is invertible.