Any Bivariate Normal.

A random pair (U,V) is said to be BIVARIATE NORMAL w/ parameters $\mu_U, \mu_V, \sigma_U, \sigma_V$, and ρ, if

$$(X,Z) := \left(\frac{U-\mu_U}{\sigma_U}, \frac{V-\mu_V}{\sigma_V} \right)$$

has the standard normal dist'n w/ correlation ρ.

Note:

$\rho_{U,V} = ?$ **Hope:** $\rho_{U,V} = \rho$ ✓

$$\rho_{U,V} = \frac{Cov[U,V]}{SD[U] \cdot SD[V]} \qquad \begin{cases} U = \mu_U + \sigma_U \cdot X \\ V = \mu_V + \sigma_V \cdot Z \end{cases}$$

$$= \frac{Cov[\mu_U + \sigma_U X, \mu_V + \sigma_V Z]}{SD[\mu_U + \sigma_U X] \cdot SD[\mu_V + \sigma_V Z]} =$$

$$= \frac{Cov[\sigma_U X, \sigma_V Z]}{SD[\sigma_U X] \cdot SD[\sigma_V Z]}$$

$$= \frac{\sigma_U \cdot \sigma_V \cdot Cov[X,Z]}{\sigma_U \cdot \underbrace{SD[X]}_{1} \cdot \sigma_V \cdot \underbrace{SD[Z]}_{1}} = \rho_{X,Z} = \rho \qquad \square$$

Problem. **Midterm and Final.**

Midterm and final scores in a large class have an (approximately) bivariate normal dist'n w/ parameters

	mean	sd
midterm	65	18
final	60	20

w/ correlation : 0.76

Q: What is the "estimated" mean final score of the students who were above the mean on the midterm?

→: Let U be the midterm score, and let V be the final score.

Our task: $\mathbb{E}[V \mid U > \mu_U] = ?$ ←

Let X and Z be U and V in std units, resp.

Our ancillary task: $\mathbb{E}[Z \mid X > 0] = ?$

Idea.

$$\mathbb{E}[Z \mid X>0] = \frac{\mathbb{E}[Z \cdot \mathbb{I}_{[X>0]}]}{\mathbb{P}[X>0]} =$$

Law of Total Probability

$$= \int_{-\infty}^{\infty} \mathbb{E}[Z \mid X=x] \cdot f_X(x \mid X>0)\, dx$$

$$= \int_{0}^{\infty} \mathbb{E}[Z \mid X=x] \cdot f_X(x \mid X>0)\, dx$$

$Z \mid X=x \sim \text{Normal}(\rho \cdot x,\ \text{var} = 1-\rho^2)$

since $Z = X \cdot \rho + Y\sqrt{1-\rho^2}$ w/ $X \perp\!\!\!\perp Y$

for $x>0$: $f_X(x \mid X>0)\, dx = \mathbb{P}[X \in (x, x+dx) \mid X>0]$

$$= \frac{\mathbb{P}[X \in (x, x+dx),\ X>0]}{\mathbb{P}[X>0]} = \frac{\mathbb{P}[X \in (x, x+dx)]}{1/2}$$

$$= 2 \cdot f_X(x)\, dx = 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\mathbb{E}[Z \mid X>0] = \int_0^\infty \rho x \cdot 2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, dx$$

$$= \frac{2\rho}{\sqrt{2\pi}} \int_0^\infty x e^{-\frac{x^2}{2}} \, dx = \left[u = \frac{x^2}{2} \Rightarrow du = \frac{1}{2} \cdot 2x\,dx \Rightarrow du = x\,dx \right]$$

$$= \frac{2\rho}{\sqrt{2\pi}} \underbrace{\int_0^\infty e^{-u} \, du}_{1} = \frac{2\rho}{\sqrt{2\pi}}$$

In this problem. $\frac{1.52}{\sqrt{2\pi}} \approx 0.6063923$

$\Rightarrow$ Our final answer: $60 + 20 \cdot \frac{1.52}{\sqrt{2\pi}} \approx 72.12785$

□