The Specifics of LDA ($p=1$).

The Choice: f_k are normal densities for each $k = 1..K$, i.e.,

$$f_k(x) = \frac{1}{\sigma_k\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu_k}{\sigma_k}\right)^2} \quad \text{for all } x \in \mathbb{R}$$

w/ μ_k being the mean and
σ_k the std deviation for class k

Additional Assumption: Homogeneity: $\sigma_k = \sigma$, $k = 1..K$

$\Rightarrow$

$$p_k(x) = \frac{\pi_k \cdot f_k(x)}{\sum_{l=1}^{K} \pi_l f_l(x)}$$

Remember: We're looking for the k for which its $p_k(x)$ is MAXIMAL over all k.

Since all $p_k(x)$ have the same denominator, it's sufficient to find the k such that

$$\pi_k \cdot f_k(x) \xrightarrow{k} \max$$

Because $\ln(\cdot)$ is increasing, the above is equivalent to:

$$\ln(\pi_k) + \ln(f_k(x)) \xrightarrow{k} \max$$

$\Leftrightarrow$

$$\ln(\pi_k) + \ln\left(\frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x-\mu_k)^2}{\sigma^2}}\right) \xrightarrow{k} \max$$

$$\Longleftrightarrow$$

$$\ln(\pi_k) - \boxed{\ln\left(\sigma\sqrt{2\pi}\right)} - \frac{(x-\mu_k)^2}{2\sigma^2} \xrightarrow{k} \max$$

constant in terms of k

$$\ln(\pi_k) \boxed{-\frac{x^2}{2\sigma^2}} + \frac{2x\mu_k}{2\sigma^2} - \frac{\mu_k^2}{2\sigma^2} \xrightarrow{k} \max$$

$$\delta_k(x) := \ln(\pi_k) + \frac{\mu_k}{\sigma^2}\cdot x - \frac{\mu_k^2}{2\sigma^2} \xrightarrow{k} \max$$

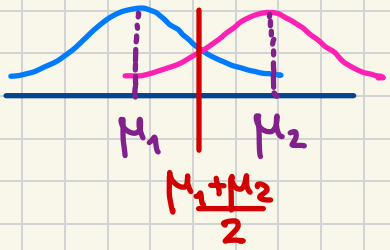
These are called **DISCRIMINANT SCORES** and they are **LINEAR** in x

Special Case:

$K = 2$, $\pi_1 = \pi_2 = \frac{1}{2}$

for $k = 1, 2$:

$$\delta_k(x) = \boxed{\ln(\pi_k)} + \frac{\mu_k}{\sigma^2}x - \frac{\mu_k^2}{2\sigma^2}$$



$$\frac{\mu_k}{\sigma^2}x - \frac{\mu_k^2}{2\sigma^2} \xrightarrow{k} \max$$

$$\mu_k\left(x - \frac{\mu_k}{2}\right) \xrightarrow{k} \max$$

IF $\mu_1 x - \frac{\mu_1^2}{2} > \mu_2 x - \frac{\mu_2^2}{2}$, THEN classify as "1".

$$(\mu_1 - \mu_2)\boxed{x} > \frac{1}{2}(\mu_1^2 - \mu_2^2) = \boxed{\frac{1}{2}}(\mu_1 - \mu_2)\boxed{(\mu_1 + \mu_2)}$$

Boundary is always $\frac{\mu_1+\mu_2}{2}$.

Bivariate Normal Random Pairs.

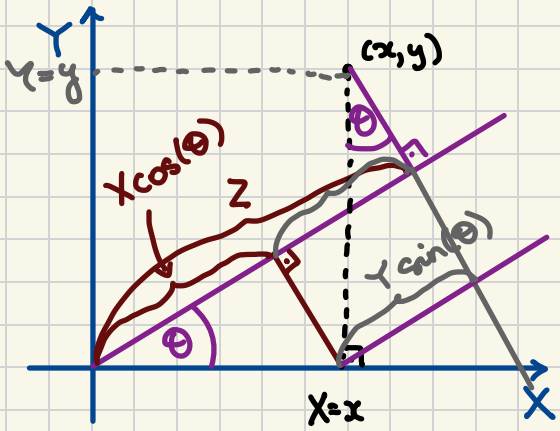
(Pitman's "Probability")

std normal density

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \text{for all } z \in \mathbb{R}$$

$Z \sim N(0,1)$

In 2D, start w/ X and Y, both $N(0,1)$ and independent.



$$\Rightarrow \boxed{Z = X\cos(\Theta) + Y\sin(\Theta)}$$

$\Rightarrow Z \sim N(\cdot,\cdot)$

$\mathbb{E}[Z] = 0$

$\mathrm{Var}[Z] = 1$

$$\boxed{Z \sim N(0,1)}$$

Q:

$$\rho_{X,Z} := \frac{\mathrm{Cov}[X,Z]}{\underbrace{SD[X]}_{1} \cdot \underbrace{SD[Z]}_{1}} = \mathrm{Cov}[X,Z]$$

$$\rho_{X,Z} = \mathbb{E}\left[(X-\mu_X)(Z-\mu_Z)\right]$$

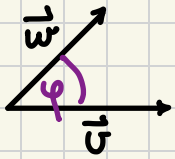
$$= \mathbb{E}[X \cdot Z] - \underbrace{\mu_X \cdot \mu_Z}_{=0}$$

$$= \mathbb{E}[X \cdot Z] = \mathbb{E}\left[X \cdot (X\cos(\Theta) + Y\sin(\Theta))\right]$$

$$= \cos(\Theta) \cdot \underbrace{\mathbb{E}[X^2]}_{=1} + \sin(\Theta)\underbrace{\mathbb{E}[XY]}_{=0}$$

$$\rho_{X,Z} = \cos(\theta)$$ □

Linear Algebra.



$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos(\varphi)$$

$$\cos(\varphi) = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \cdot \|\vec{w}\|}$$

Special Cases:

$\theta = 0 \Rightarrow Z = X$

$\theta = \frac{\pi}{2} \Rightarrow Z = Y$

So, X and Z are INDEPENDENT.

$\theta = \pi \Rightarrow Z = -X$

In general: $-1 \leq \rho \leq 1$,

there exists an angle $\boxed{\theta = \arccos(\rho)}$

s.t. $\rho_{X,Z} = \rho$

Alternative: $\boxed{Z = \rho \cdot X + \sqrt{1-\rho^2} \cdot Y}$ ←

w/ X and Y independent and $N(0,1)$

Joint Density:

$$f_{X,Z}(x,z) = \frac{1}{2\pi(1-\rho^2)} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xz + z^2)\right)$$

Marginal Dist'ns: $X \sim N(0,1)$, $Z \sim N(0,1)$

Conditional Dist'ns: Given $X = x$, $Z \sim N(\rho \cdot x, \text{var} = 1-\rho^2)$

Given $Z = z$, $X \sim N(\rho \cdot z, \text{var} = 1-\rho^2)$