The University of Texas at Austin
In-Class Work 13

M339D Introduction to Financial Mathematics

October 21, 2026

Risk-Neutral Probability.

In all the problems below, the stock pays no dividends and is modeled using a one-period binomial tree over one year. The risk-free rate is continuously compounded.

Problem 13.1: *Consider a non-dividend paying stock whose current price is \$95 per share. You model the evolution of this stock price over the following year using a one-period binomial tree under the assumption that the stock price can be either \$120, or \$75 in one year.*

The continuously compounded, risk-free rate is 0.06.

A ***straddle*** *consists of a long call and a long otherwise identical put. Consider a \$100-strike, one-year European straddle on the above stock. What is the straddle's price consistent with the above stock-price model?*

a. *About \$10*
b. *About \$10.83*
c. *About \$15.45*
d. *About \$20.84*
e. *None of the above.*

Solution: ***(d)***
The risk-neutral probability of an up movement is

$$p^* = \frac{95e^{0.06} - 75}{120 - 75} = 0.575.$$

So, the price of our straddle is

$$V(0) = e^{-0.06}[0.575 \times (120 - 100) + (1 - 0.575) * (100 - 75)] = 20.8366.$$

Problem 13.2: *The current price of a stock is \$100 per share. In one year, the stock price will be either \$130 or \$85.*

The continuously compounded, risk-free rate is 0.02.

A one-year contract pays the larger of the payoff of a \$105-strike European call and the payoff of a \$105-strike European put. What is the contract's price?

a. *About \$9.27*
b. *About \$12.19*
c. *About \$21.46*
d. *About \$21.89*

e. None of the above.

Solution: ***(c)***
The contract's payoff is $|S(1) - 105|$*, i.e., the payoff of a \$105-strike straddle. The risk-neutral probability of an up movement is*

$$p^* = \frac{100e^{0.02} - 85}{130 - 85} = 0.3782.$$

So, the contract's price is

$$V(0) = e^{-0.02}[0.3782 \times (130 - 105) + (1 - 0.3782) \times (105 - 85)] = 9.27 + 12.19 = 21.46.$$

Problem 13.3: *The current price of a stock is \$20 per share. In one year, the stock price will be either \$26 or \$16.*

The continuously compounded, risk-free rate is 0.04*.*

A one-year contract pays $(S(1) - 20)^2$*. What is the contract's price?*

a. 0
b. About \$24.63
c. About \$24.97
d. About \$25.63
e. None of the above.

Solution: ***(b)***
The risk-neutral probability of an up movement is

$$p^* = \frac{20e^{0.04} - 16}{26 - 16} = 0.4816.$$

So, the contract's price is

$$V(0) = e^{-0.04}[0.4816 \times 36 + (1 - 0.4816) \times 16] = 24.63.$$

Remark 13.4*: The risk-neutral variance of* $S(1)$ *is* $p^*(1 - p^*) \times 10^2 = 24.97$*, and*

$$e^{0.04}V(0) = \text{Var}^*[S(1)] + (E^*[S(1)] - 20)^2 = 24.97 + 0.67 = 25.63.$$

Problem 13.5: *The current price of a stock is \$50 per share. In one year, the stock price will be either* S_u *or \$40.*

The continuously compounded, risk-free rate is 0.05*.*

A \$55-strike, one-year European call on the stock trades for \$2.99. What is S_u*?*

a. About \$55
b. About \$58
c. About \$60

d. About \$62.50
e. None of the above.

Solution: ***(c)***
The risk-neutral probability of an up movement is

$$p^* = \frac{50e^{0.05} - 40}{S_u - 40} = \frac{12.5636}{S_u - 40}.$$

So, the call's price is

$$e^{-0.05} \cdot \frac{12.5636}{S_u - 40} \cdot (S_u - 55) = 2.99.$$

Solving, we get $S_u \approx 60$.

Problem 13.6: *The current price of a stock is \$40 per share. In one year, the stock price will be either \$50 or \$32.*

The continuously compounded, risk-free rate is 0.03.

Analyst A believes the probability of an up movement is 0.9. *Analyst B believes it is* 0.1. *Both analysts price a \$40-strike, one-year European call using the above model. Which statement is true?*

a. Analyst A's price is higher.
b. Analyst B's price is higher.
c. Both prices equal about \$4.97.
d. Both prices equal about \$5.12.
e. None of the above.

Solution: ***(c)***
The real-world probability does not enter the price. The risk-neutral probability of an up movement is

$$p^* = \frac{40e^{0.03} - 32}{50 - 32} = 0.5121.$$

So, the call's price is

$$V_C(0) = e^{-0.03}[0.5121 \times (50 - 40)] = 4.97.$$

> ***Remark 13.7**: Under A's beliefs, the stock's expected log-return is about* 18.6% *under B's, it is about* -16.8%. *Yet, the call price is the same.*

Problem 13.8: *The current price of a stock is \$80 per share. In one year, the stock price will be either \$100 or \$70.*

The continuously compounded, risk-free rate is 0.04.

The market quotes an \$85-strike, one-year European call at \$6.37 and an otherwise identical put at \$7.50. Which statement is true?

a. The call is underpriced. An arbitrage portfolio yields about \$0.54 today.
b. The put is underpriced. An arbitrage portfolio yields about \$0.54 today.
c. The call is overpriced. An arbitrage portfolio yields about \$0.54 today.
d. The put is underpriced. An arbitrage portfolio yields about \$1.67 today.
e. None of the above.

Solution: ***(b)***
The risk-neutral probability of an up movement is

$$p^* = \frac{80e^{0.04} - 70}{100 - 70} = 0.4422.$$

So, the no-arbitrage prices are

$$C(0) = e^{-0.04}[0.4422 \times (100 - 85)] = 6.37, \quad P(0) = e^{-0.04}[(1 - 0.4422) \times (85 - 70)] = 8.04.$$

Thus, the put is underpriced. Its replicating portfolio is

$$\Delta = \frac{0 - 15}{100 - 70} = -0.5, \quad B = e^{-0.04}\frac{100 \times 15 - 70 \times 0}{100 - 70} = 48.04.$$

Buy the put for \$7.50, buy 0.5 *shares, and borrow \$48.04. The net cash flow today is* $48.04 - 40 - 7.50 = 0.54$*, and the payoff at time* 1 *is zero in both states.*

Problem 13.9: *The current price of a stock is \$50 per share. In one year, the stock price will be either \$60 or \$45.*

The continuously compounded, risk-free rate is 0.03*.*

You cannot trade the stock. You can only trade a \$50-strike, one-year European call and the risk-free asset. Which portfolio replicates the \$50-strike, one-year European put?

a. Long 0.5 *calls and a risk-free investment paying \$5 at time* 1*.*
b. Short 0.5 *calls and a risk-free investment paying \$5 at time* 1*.*
c. Short 1 *call and a risk-free investment paying \$50 at time* 1*.*
d. Short 1.5 *calls and a risk-free investment paying \$45 at time* 1*.*
e. None of the above.

Solution: ***(b)***
The call pays 10 *in the up state and* 0 *in the down state. The put pays* 0 *in the up state and* 5 *in the down state. If we hold* a *calls and a risk-free investment paying* b *at time* 1*, they must satisfy these equations to replicate a put*

$$10a + b = 0, \quad b = 5 \quad \Rightarrow \quad a = -0.5.$$

***Remark 13.10**: If we had been asked to price the put, we could price it directly or via the call price. With the risk-neutral probability of* $p^* = \frac{50e^{0.03} - 45}{60 - 45} = 0.4348$*, the call's price is* $V_C(0) = e^{-0.03}(0.4348 \times 10) = 4.22$*. So,*

$$V_P(0) = -0.5 \times 4.22 + 5e^{-0.03} = 2.74.$$

Similarly, the stock can be remplcated replicated by 1.5 *calls and a risk-free investment paying \$45 at time* 1*. Note that* $1.5 \times 4.22 + 45e^{-0.03} = 50$*.*

Problem 13.11: *The current price of a stock is \$30 per share. In one year, the stock price will be either \$33 or \$27.*

The continuously compounded, risk-free rate is 0.12*.*

Consider the strategy: short sell one share and invest the proceeds at the risk-free rate for one year. What is the smallest possible profit of this strategy at time 1*?*

a. 0
b. About \$0.82
c. About \$3.00
d. About \$3.82
e. None of the above.

Solution: ***(b)***
Since $30e^{0.12} = 33.82 > 33$*, we get* $p^* > 1$*, so there is no well-defined risk-neutral probability and the model admits arbitrage.*

At time 1*, the investment is worth \$33.82, and buying back the share costs at most \$33. So, the profit is at least* $33.82 - 33 = 0.82$ *(in the up state) and equals* $33.82 - 27 = 6.82$ *in the down state.*