The University of Texas at Austin
In-Class Work 11

M339D Introduction to Financial Mathematics

October 16, 2026

The Binomial Asset Pricing Model.

Problem 11.1. In the setting of the one-period binomial model, denote by i the **effective** interest rate per period. Let u denote the "up factor" and let d denote the "down factor" in the stock-price model. If

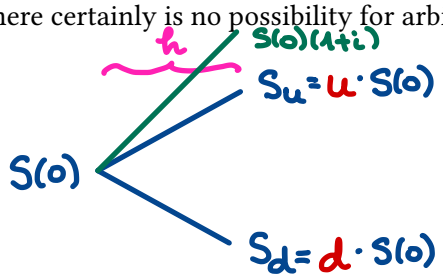
$$d < u \leq 1 + i$$

← Propose an arbitrage portfolio?

e.g., short the stock

Verify this @ home.

then there certainly is no possibility for arbitrage. *True or false? Why?*



No-Arbitrage Condition

$d < e^{rh} < u$ ←

$d < 1+i < u$

This is the fixed statement!

Problem 11.2. In our usual notation, does this parameter choice create a binomial model with an arbitrage opportunity?

$u = 1.18, \quad d = 0.87, \quad r = 0.05, \quad \delta = 0, \quad h = 1/4$

No, it doesn't ☺

→: Check: $d = 0.87 \overset{?}{<} e^{rh} = e^{0.05 \cdot (1/4)} \overset{?}{<} u = 1.18$

$e^{0.0125}$

$\overset{??}{\approx} 1.0125$

Taylor Expansion of e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

Q: What if $\tilde{d} = 1.01$?

This is still ok!

Realized Returns.

$$R(t,t+s) := \ln\left(\frac{S(t+s)}{S(t)}\right)$$

$$S(t+s) = S(t)\,e^{R(t,t+s)}$$

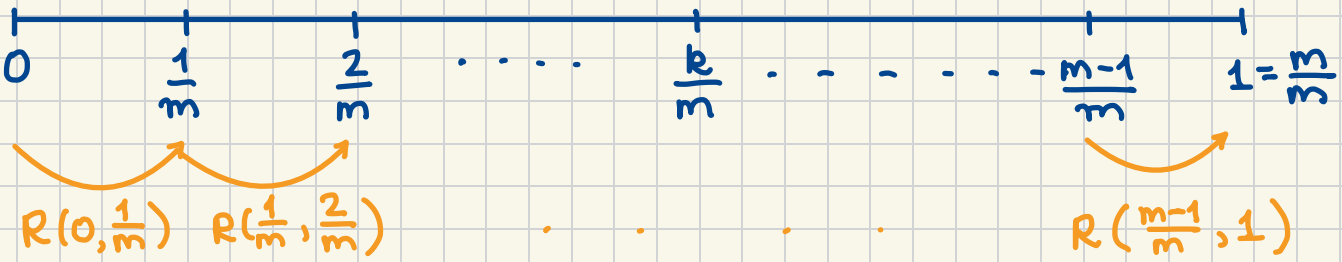
Heuristics.

$T = 1$. Q: $\mathrm{Var}[R(0,1)] = \sigma^2$

$h_m = \frac{1}{m}$ (of a year)

Q: What is the volatility for the time period of length h_m?

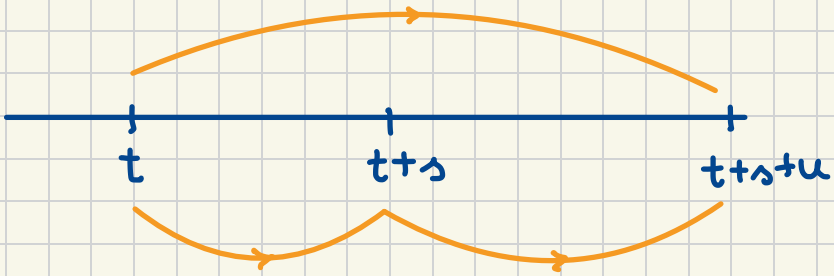
$$\sigma_{h_m} = ?$$ ←



$R\left(\frac{k-1}{m},\frac{k}{m}\right)$ for $k = 1,\dots,m$ are all RANDOM VARIABLES.

We make the following assumptions:

- the returns over disjoint intervals are independent;
- all the returns over the same length intervals are identically distributed.



$$R(t, t+s+u) = \ln\left(\frac{S(t+s+u)}{S(t)}\right)$$

$$= \ln\left(\frac{S(t+s)}{S(t)} \cdot \frac{S(t+s+u)}{S(t+s)}\right)$$

$$= \ln\left(\frac{S(t+s)}{S(t)}\right) + \ln\left(\frac{S(t+s+u)}{S(t+s)}\right)$$

$$= R(t, t+s) + R(t+s, t+s+u)$$

Hence, realized returns are **additive**.

$\Rightarrow$

$$R\left(0, \tfrac{1}{m}\right) + R\left(\tfrac{1}{m}, \tfrac{2}{m}\right) + \cdots + R\left(\tfrac{m-1}{m}, 1\right) = R(0,1)$$

$$\mathrm{Var}\left[R\left(0, \tfrac{1}{m}\right) + \cdots + R\left(\tfrac{m-1}{m}, 1\right)\right] = \mathrm{Var}[R(0,1)] = \sigma^2$$

(independence)

$$\mathrm{Var}\left[R\left(0, \tfrac{1}{m}\right)\right] + \cdots + \mathrm{Var}\left[R\left(\tfrac{m-1}{m}, 1\right)\right] = \sigma^2$$

(identically dist'd)

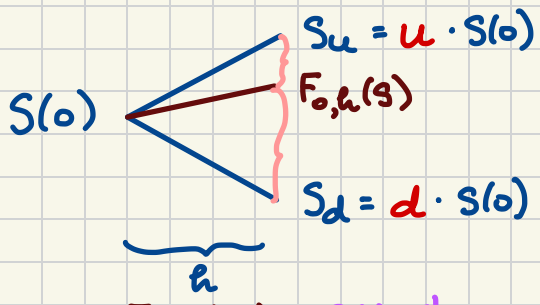
$$m \cdot \mathrm{Var}\left[R\left(0, \tfrac{1}{m}\right)\right] = \sigma^2$$

$$\sigma_{hm}^2 = \frac{\sigma^2}{m} \quad \Rightarrow \quad \boxed{\sigma_{hm} = \sigma\sqrt{\frac{1}{m}} = \sigma\sqrt{hm}}$$

We generalize this identity to arbitrary lengths h:

$$\boxed{\sigma_h = \sigma\sqrt{h}}$$

Forward Binomial Trees [revisited].



No-Arbitrage Cond'n:

$$d < e^{rh} < u$$

Recall: $F_{0,h}(S) = S(0)e^{rh}$

$$S_u := F_{0,h}(S)\, e^{\sigma\sqrt{h}} = S(0)e^{rh} \cdot e^{\sigma\sqrt{h}} = S(0)e^{rh+\sigma\sqrt{h}} = u$$

$$S_d := F_{0,h}(S)\, e^{-\sigma\sqrt{h}} = S(0)e^{rh} \cdot e^{-\sigma\sqrt{h}} = S(0)e^{rh-\sigma\sqrt{h}} = d$$

Q: Do u and d satisfy the no-arbitrage condition? ✓

$\longrightarrow$:

$$d < e^{rh} < u$$

$$e^{rh-\sigma\sqrt{h}} < e^{rh} < e^{rh+\sigma\sqrt{h}}$$

$$e^{-\sigma\sqrt{h}} < 1 < e^{\sigma\sqrt{h}} \qquad \sigma, h > 0$$

□

Q: What is $\frac{S_u}{S_d} = ?$

$\longrightarrow$:

$$\frac{S_u}{S_d} = \frac{S(0)e^{rh} \cdot e^{\sigma\sqrt{h}}}{S(0)e^{rh} \cdot e^{-\sigma\sqrt{h}}} = e^{2\sigma\sqrt{h}}$$

□