The University of Texas at Austin
Homework Assignment 11

Introduction to Financial Mathematics

October 31, 2026

Instructions: Provide your complete solution to the following problems. Final answers only, without appropriate justification, will receive zero points even if correct.

The normal distribution.

Problem 11.1. (2 points) A linear combination of two jointly normally distributed random variables is always also normally distributed. Assume that a constant is also considered normally distributed with variance zero. *True or false?*

Solution. TRUE

Problem 11.2. (5 points) If the random variable X is standard normal, then the distribution function of the random variable $Y = |X|$ equals

$$F_Y(a) = 2N(a) - 1 \text{ for every } a \geq 0.$$

True or false? Why?

Solution. TRUE
For every $a \geq 0$,

$$\begin{aligned} F_Y(a) &= \mathbb{P}[Y \leq a] \\ &= \mathbb{P}[|X| \leq a] \\ &= \mathbb{P}[-a \leq X \leq a] \\ &= \mathbb{P}[X \leq a] - \mathbb{P}[X < -a] \\ &= \mathbb{P}[X \leq a] - (1 - \mathbb{P}[X \geq -a]) \\ &= \mathbb{P}[X \leq a] - (1 - \mathbb{P}[X \leq a]) \\ &= 2N(a) - 1. \end{aligned}$$

Problem 11.3. (15 points) Let Z be a standard normal random variable. Find the following probabilities:

- $\mathbb{P}[-1.33 < Z \leq 0.24]$
- $\mathbb{P}[0.49 < |Z|]$
- $\mathbb{P}\left[Z^4 < 0.0256\right]$
- $\mathbb{P}\left[e^{2Z} < 2.25\right]$
- $\mathbb{P}\left[\frac{1}{Z} < 2\right]$

Solution.

- $$\begin{aligned}\mathbb{P}[-1.33 < Z \leq 0.24] &= \mathbb{P}[Z \leq 0.24] - \mathbb{P}[Z \leq -1.33] = \mathbb{P}[Z \leq 0.24] - (1 - \mathbb{P}[Z \leq 1.33]) \\ &= 0.5948 - 1 + 0.9082 = 0.503\end{aligned}$$
- $$\begin{aligned}\mathbb{P}[0.49 < |Z|] &= \mathbb{P}[Z < -0.49] + \mathbb{P}[0.49 < Z] = 2\mathbb{P}[Z > 0.49] \\ &= 2(1 - \mathbb{P}[Z \leq 0.49]) = 2(1 - 0.6879) = 0.6242\end{aligned}$$
- $$\begin{aligned}\mathbb{P}[Z^4 < 0.0256] &= \mathbb{P}\left[|Z| < \sqrt[4]{0.0256}\right] = \mathbb{P}[|Z| < 0.4] = \mathbb{P}[Z < 0.4] - \mathbb{P}[Z < -0.4] \\ &= 2\mathbb{P}[Z < 0.4] - 1 = 2(0.6554) - 1 = 0.3108\end{aligned}$$
- $$\mathbb{P}[e^{2Z} < 2.25] = \mathbb{P}[2Z < \ln(2.25)] = \mathbb{P}[Z < 0.5\ln(2.25)] \approx \mathbb{P}[Z \leq 0.41] = 0.6591$$
- $$\begin{aligned}\mathbb{P}\left[\frac{1}{Z} < 2\right] &= \mathbb{P}\left[\frac{1}{Z} < 0\right] + \mathbb{P}\left[0 < \frac{1}{Z} < 2\right] \\ &= \mathbb{P}[Z < 0] + \mathbb{P}[Z > 0.5] \\ &= 0.5 + (1 - \mathbb{P}[Z \leq 0.5]) = 0.5 + (1 - 0.6915) = 0.8085.\end{aligned}$$

Problem 11.4. (10 points) Two laser pointers are used to measure the length ℓ of a building. The error made by the less precise laser pointer is normally distributed with mean 0 and standard deviation 0.0144ℓ. The error made by the more precise laser pointer is normally distributed with mean 0 and standard deviation 0.0036ℓ. The errors from the two laser pointers are independent of each other. Calculate the probability that the average value of the two measurements is within 0.001ℓ of the true length ℓ of the building.

Solution. Let the error of the measurement from the first laser pointer be denoted by ε_1 and let the error of the measurement from the second laser pointer be denoted by ε_2. According to our modelling assumptions

$$\begin{aligned}\varepsilon_1 &\sim Normal(mean = 0, sd = 0.0144\ell), \\ \varepsilon_2 &\sim Normal(mean = 0, sd = 0.0036\ell).\end{aligned}$$

Let the average error be $\varepsilon = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$. Then, ε is normally distributed with mean 0 and variance

$$Var[\varepsilon] = \frac{1}{4}(Var[\varepsilon_1] + Var[\varepsilon_2]) = \frac{1}{4}((0.0144\ell)^2 + (0.0036\ell)^2).$$

So,

$$\varepsilon \sim Normal(mean = 0, sd = 0.0074\ell).$$

We need to find the probability $\mathbb{P}[|\varepsilon| < 0.001\ell]$. We have

$$\begin{aligned}\mathbb{P}[|\varepsilon| < 0.001\ell] &= \mathbb{P}[-0.001\ell < \varepsilon < 0.001\ell] \\ &= \mathbb{P}\left[\frac{-0.001\ell - 0}{0.0074\ell} < \frac{\varepsilon - 0}{0.0074\ell} < \frac{0.001\ell - 0}{0.0074\ell}\right] = \mathbb{P}[-0.1351 < Z < 0.1351]\end{aligned}$$

where $Z \sim N(0, 1)$. So,

$$\mathbb{P}[|\varepsilon| < 0.001\ell] = 2N(0.1351) - 1 \approx 2N(0.14) - 1 = 2(0.5557) - 1 = 0.1114.$$

Problem 11.5. (10 points) An astronomical instrument measures the distance d to a far-away planet. You know that the instrument is calibrated so that its measurement error is normally distributed, centered around zero, and with variance $(0.0001d)^2$. The different measurements using this same instrument are assumed to be independent. How many independent measurements would you have to perform so that your average is within $10^{-6}d$ with probability 99%?

Solution. For every $i \in \mathbb{N}$, let ε_i denote the error of the i^{th} measurement. Then, with n being the number of measurements, the average error $\varepsilon = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i$ is normally distributed with mean 0 and with variance

$$Var[\varepsilon] = Var\left[\frac{1}{n}(\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n)\right] = \frac{1}{n^2} Var[\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_n].$$

Due to the fact that different measurements are independent, we get

$$Var[\varepsilon] = \frac{1}{n^2}(Var[\varepsilon_1] + Var[\varepsilon_2] + ... + Var[\varepsilon_n]) = \frac{1}{n^2}(n)(0.0001d)^2 = \frac{(0.0001d)^2}{n}.$$

The probability we are given is

$$\begin{aligned} \mathbb{P}[|\varepsilon| < 10^{-6}d] = 0.99 \quad &\Rightarrow \quad \mathbb{P}[-10^{-6}d < \varepsilon < 10^{-6}d] = 0.99 \\ &\Rightarrow \quad \mathbb{P}\left[-\frac{10^{-6}d - 0}{\frac{0.0001d}{\sqrt{n}}} < \frac{\varepsilon - 0}{\frac{0.0001d}{\sqrt{n}}} < \frac{10^{-6}d - 0}{\frac{0.0001d}{\sqrt{n}}}\right] = 0.99 \\ &\Rightarrow \quad \mathbb{P}\left[-\frac{10^{-6}\sqrt{n}}{10^{-4}} < Z < \frac{10^{-6}\sqrt{n}}{10^{-4}}\right] = 0.99 \end{aligned}$$

where $Z \sim N(0,1)$. So, we have to find a condition for n in

$$\mathbb{P}\left[-\frac{\sqrt{n}}{100} < Z < \frac{\sqrt{n}}{100}\right] = 0.99 \quad \Rightarrow \quad 2N\left(\frac{\sqrt{n}}{100}\right) - 1 = 0.99 \quad \Rightarrow \quad N\left(\frac{\sqrt{n}}{100}\right) = 0.995.$$

Using R or the standard normal tables, we get

$$\frac{\sqrt{n}}{100} \geq 2.576 \quad \Rightarrow \quad n \geq (257.6)^2 \quad \Rightarrow \quad n \geq 66358.$$

Problem 11.6. (8 points) The profit of a certain company is modelled using a normal distribution with mean $1,000,000$ and standard deviation $400,000$. Given that the profit is positive, what is the probability that it is below $1,200,000$?

Solution. Let X denote the profit random variable. Then, we need to calculate

$$\mathbb{P}[X < 1200000 \,|\, X > 0] = \frac{\mathbb{P}[0 < X < 1200000]}{\mathbb{P}[X > 0]} = \frac{\mathbb{P}[X < 1200000] - \mathbb{P}[X \leq 0]}{\mathbb{P}[X > 0]}.$$

The given model for X is

$$X \sim Normal(mean = 1000000, sd = 400000).$$

First, let's calculate the probability that the profit is below zero

$$\mathbb{P}[X \leq 0] = \mathbb{P}\left[\frac{X - 1000000}{400000} \leq \frac{0 - 1000000}{400000}\right] = \mathbb{P}[Z \leq -2.5] = 0.0062.$$

Above, we denoted by Z a standard normal random variable. So, $\mathbb{P}[X > 0] = 0.9938$. Next, we calculate the probability that the profit is below $1,200,000$.

$$\mathbb{P}[X \leq 1200000] = \mathbb{P}\left[\frac{X - 1000000}{400000} \leq \frac{1200000 - 1000000}{400000}\right] = \mathbb{P}[Z \leq 0.5] = 0.6915.$$

Finally, our answer is

$$\mathbb{P}[X < 1200000 \,|\, X > 0] = \frac{0.6915 - 0.0062}{0.9938} = 0.6895754.$$